Linear Algebra I

10/10/2022

You are **NOT** allowed to use any type of calculators.

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1 Systems of linear equations $(4 + 1 + 10 + 5 = 20 \text{ pts})$

Two groups of students attend lectures in the same classroom. The first group consists of p students and the second group consists of q students. In the classroom, there are r rows of seats. When the first group follows a lecture, there are 6 students sitting at each row except the last row which is occupied only by 4 students. When the second group follows another lecture, there are 5 students sitting at each row except the last row which is occupied by 6 students. When both groups follow yet another lecture, there are 10 students sitting at each row and 10 more standing.

(a) Find a system of linear equations in the unknowns p, q, and r describing the above scenario.

(b) Write down the augmented matrix.

(c) By performing elementary row operations, put the augmented matrix into **reduced** row echelon form.

(d) Determine the solution set.

2 Matrix multiplication (20 pts)

Let λ be a scalar. Consider the matrix

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

By mathematical induction on k, show that

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{(k-1)k}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

for all $k \geqslant 2$.

3 Nonsingular matrices (5 + 12 + 8 = 25 pts)

Let a be a real number and $x \in \mathbb{R}^n$ be a vector with $x^T x = 1$. Consider

$$M(a) = I_n + axx^T.$$

(a) Let a, b be given. Show that there exists a real number c such that $M(a)M(b) = M(c)$.

(b) Show that $M(a)$ is nonsingular if and only if $a \neq -1$.

(c) Find the inverse of $M(a)$ for $a \neq -1$.

4 Determinants (20 + 5 = 25 pts)

Let a, b, c, d be scalars. Consider the matrix

$$M = \begin{bmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{bmatrix}.$$

(a) Find $\det M$.

(b) Suppose that $b = c = d = 1$. Determine all values of a such that M is **singular**.

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10 pts free